FINITE SIMPLE GROUPS OF LIE TYPE AS EXPANDERS

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Dedicated to the memory of Beth Samuels who is deeply missed

1. Introduction

A finite k-regular graph X, $k \in \mathbb{N}$, is called an **ε-expander** ($0 < \varepsilon \in \mathbb{R}$), if for every subset of vertices A of X, with $|A| \leq \frac{1}{2}|X|$, $|\partial A| \geq \varepsilon |A|$ where $\partial A = \{y \in X |$ distance $(y, A) = 1\}$.

The main goal of this paper is to prove:

Theorem 1.1. *There exist $k \in \mathbb{N}$ and $0 < \varepsilon \in \mathbb{R}$, such that if G is a finite simple group of Lie type, but not a Suzuki group, then G has a set of k generators S for which the Cayley graph $Cay(G; S)$ is an ε-expander.*

By abuse of the language, we will say that these groups are uniform expanders or expanders uniformly.

Theorem 1.1 is new only for groups of small Lie rank: In [K1], Kassabov proved that the groups

$$\{\mathrm{SL}_n(q) | 3 \leq n \in \mathbb{N}, \quad q \text{ a prime power}\}$$

are uniform expanders. Nikolov [N] proved that every **classical** group is a bounded product of $\mathrm{SL}_n(q)$'s (with possible $n = 2$, but the proof shows that if the Lie rank is sufficiently high, say ≥ 14, one can use $\mathrm{SL}_n(q)$ with $n \geq 3$). Bounded products of uniform expanders are uniform expanders (see Corollary 2.2 below). Thus together, their results cover all classical groups of high rank. So, our Theorem is new for classical groups of small ranks as well as for the families of exceptional groups of Lie type.

Theorem 1.1 gives the last step of the result conjectured in [BKL] and announced in [KLN]:

Theorem 1.2 ([KLN])**.** *All non-abelian finite simple groups, with the possible exception of the Suzuki groups, are uniform expanders.*

By the classification of the finite simple groups, Theorem 1.1 covers all the simple groups except of finitely many sporadic groups (for which the theorem is trivial) and the alternating groups. The fact that Theorem 1.2 holds for the alternating and the symmetric groups is a remarkable result of Kassabov [K2].

The main new family covered by our method is $\{\mathrm{PSL}_2(q) | q \text{ prime power}\}$. Unlike the results mentioned previously ([K1, K2]) whose proofs used ingenious, but relatively elementary methods, the proof for $\mathrm{PSL}_2(q)$ will use some deep results from the theory of automorphic forms. In particular, it will appeal to Selberg $\lambda_1 \geq \frac{3}{16}$ Theorem ([Se], see also [Lu, Chap. 4]) and Drinfeld solution to the characteristic p Ramanujan conjecture ([Dr]).

For its importance, let us single it out as:

Theorem 1.3. *The family* $\{\mathrm{PSL}_2(q) | q$ *prime power*$\}$ *forms a family of uniform expanders.*

Let us mention right away that Theorem 1.3 was known before for several subfamilies; e.g. for $\{\mathrm{PSL}_2(p) | p \text{ prime}\}$ (see [Lu, Chap. 4]] or $\{\mathrm{PSL}_2(p^r) | p \text{ a fixed prime and } r \in \mathbb{N}\}$ ([Mo]). The main novelty is to make them expanders uniformly for all p and all r. To this end we will use the representation theoretic reformulation of the expanding property (see §2) as well as the new **explicit** constructions of Ramanujan graphs in [LSV2] as special cases of Ramanujan complexes. We stress that the explicit construction there is crucial for our method and not only the theoretical construction of [LSV1]. This will be shown in §3.

The case of SL_2 is a key step for the other groups of Lie type: A result of Hadad ([H1], which is heavily influenced by Kassabov [K1]) enables one to deduce SL_n $(n \geq 2)$ from SL_2. Then in §4, we use a model theoretic argument to show that simple groups of Lie type of bounded rank (including the exceptional families except of the Suzuki groups) are bounded products of SL_2's. Together with Nikolov's result mentioned above, Theorem 1.1 is then fully deduced.

The Suzuki groups have to be excluded as they do not contain a copy of $(\mathrm{P})\mathrm{SL}_2(q)$ for any q, but we believe that Theorem 1.2 holds for them as well.

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2. Representation theoretic reformulation

It is well known (cf. [Lu, Chap. 4]) that expanding properties of Cayley graphs $Cay(G;S)$ can be reformulated in the language of the representation theory of G. For our purpose we will need to consider also cases for which S is not of bounded size, in spite of the fact that our final result deals with bounded S. We therefore need a small extension of some standard results, for which we need some notation:

The normalized adjacency matrix of a connected k-regular graph X is defined to be $\triangle = \frac{1}{k}A$ where A is the adjacency matrix of X. The eigenvalues of $\triangle$ are in the interval $[-1,1]$. The largest eigenvalue in absolute value in $(-1,1)$ is denoted $\lambda(X)$.

For a group G, a set of generators S and $\alpha > 0$, we denote by $I(\alpha, G, S)$ the statement:

For every unitary representation (V,ρ) of G, every $v \in V$ and every $0 < \delta \in \mathbb{R}$, if $\|\rho(s)v - v\| < \delta$ for each $s \in S$, then $\|\rho(g)v - v\| < \alpha\delta$ for every $g \in G$, (i.e., a vector v which is "S-almost invariant" is also "G-almost invariant".)

Note that the statement $I(\alpha, G, S)$ refers to all the unitary representations of G, whether they have invariant vectors or not.

Proposition 2.1. (i) *For every $\alpha > 0$ there is $\varepsilon = \varepsilon(\alpha) > 0$ such that if G is a finite group, S a set of generators and $I(\alpha, G, S)$ holds, then $Cay(G,S)$ is an ε-expander.*

(ii) *For every $\eta > 0$, there exists $\alpha = \alpha(\eta)$ such that if G is a finite group with a set of generators S, and $\lambda(Cay(G,S)) < 1 - \eta$, then $I(\alpha, G, S)$ holds.*

(iii) *If $k = |S|$ is bounded then the implications in* (i) *can be reversed. (So $Cay(G,S)$ is expander iff every "S-almost invariant" vector is also "G-almost invariant".)*

Proof. We note first that property $I(\alpha, G, S)$ implies that there exists $\beta = \beta(\alpha) > 0$, such that for every unitary representation (V,ρ) of G without a non-zero invariant vector, and every $v \in V$ with $\|v\| = 1$, $\|\rho(s)v - v\| \geq \beta$ for some $s \in S$. Indeed, take $\beta < \frac{1}{2\alpha}$ and so if $\|\rho(s)v - v\| < \beta$ for every $s \in S$, then $I(\alpha, G, S)$ implies that $\|\rho(g)v - v\| < \frac{1}{2}$ for every $g \in G$. This implies that $\overline{v} = \frac{1}{|G|}\sum_{g \in G} \rho(g)v$ which is clearly a G-invariant vector, is non-zero since $\|\overline{v} - v\| < \frac{1}{2}$. This contradicts our assumption that V does not contain an invariant vector.

Altogether, $I(\alpha, G, S)$ implies the usual "property T" formulation and so the standard proof of Proposition 3.3.1 of [Lu] applies to deduce that $Cay(G, S)$ is an ε-expander for some $\varepsilon = \varepsilon(\beta(\alpha))$. This proves (i). The proof of (ii) is also a small modification of the standard equivalences (see [Lu, Theorem 4.3.2]): As it is well known, a normalized eigenvalue gap (i.e. $\lambda(Cay(G, S)) < 1 - \eta$) implies an "average expanding", i.e. if (V, ρ) does not contain an invariant vector, then

$$(*) \qquad \frac{1}{|S|} \sum_{s \in S} \|\rho(s)v - v\| \geq \eta' \|v\|$$

(where η' depends only on η). Note, that when S is unbounded, this is a stronger property than "expanding" which gives that for one $s \in S$, $\|\rho(s)v - v\| \geq \eta'' \|v\|$ (for $\eta'' = \eta''(\eta)$). Now, assume (V, ρ) is an arbitrary unitary representation space of G and $v \in V$, of norm one, is δ-invariant under S for some $\delta < \eta'$. Then $(*)$ implies that a large portion of v is in the space V^G of G-fixed points. Hence v is G-almost invariant as needed.

Part (iii) is just the standard equivalences as in [Lu, Theorem 4.3.2]. □

An easy corollary of Proposition 2.1 is that 'bounded products of expanders are expanders' or in a precise form:

Corollary 2.2. Let G be a finite group and $G_i, i = 1, \ldots, \ell$, a family of subgroups of G, each comes with a set of generators $S_i \subseteq G_i, i = 1, \ldots, \ell$, with $|S_i| \leq r$. Assume $G = G_1 \cdot \ldots \cdot G_\ell$, i.e., every $g \in G$ can be written as $g = g_1 g_2 \ldots g_\ell$, with $g_i \in G_i$. If all $Cay(G_i; S_i)$ are δ-expanders, then $Cay(G; S)$ is an ε-expander for $S = \bigcup_{i=1}^{\ell} S_i$ and ε which depends only on δ and ℓ.

Proof. If (V, ρ) is a unitary representation of G, and $v \in V$ is a vector which is almost invariant under S, then it is almost invariant under each of the subgroups G_i (by (2.1)(iii)) and as G is a product of them, it is also almost invariant by G. Now use (2.1)(i) to deduce the Corollary. □

Let us mention here another fact that will be used freely later. The following Proposition is a special case of a more general result in [H2]:

Proposition 2.3. *Let $\{G_i\}_{i \in I}$ be a family of perfect finite groups (i.e. $[G_i, G_i] = G_i$) with sets of generators S_i. Assume $\pi_i : \tilde{G}_i \to G_i$ is a central perfect cover of G_i and $\tilde{S}_i \subset \tilde{G}_i$ a subset for which $\pi(\tilde{S}_i) = S_i$*

and $|\tilde{S}_i| = S_i$. *If* $Cay(G_i, S_i)$ *are uniformly expanders, then so are* $Cay(\tilde{G}_i, \tilde{S}_i)$.

The Proposition shows that proving uniform expanding for finite simple groups or for their central extensions is the same problem. So a family of groups of the form $\mathrm{PSL}_d(q)$ are expanders iff $\mathrm{SL}_d(q)$ are.

3. SL_2: Proof of Theorem 1.3

The goal of this section is to show that all the groups $\{\mathrm{SL}_2(q) | q \text{ prime power}\}$ (and hence also $\mathrm{PSL}_2(q)$) are uniformly expanders. Let us recall

Theorem 3.1. *The Cayley graphs* $Cay(\mathrm{PSL}_2(p); \left\{A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\})$, *for* p *prime, are 3-regular uniform expanders.*

For a proof, see [Lu, Theorem 4.4.2]. The proof uses Selberg Theorem $\lambda_1 (\Gamma(m)\backslash\mathbb{H}^2) \geq \frac{3}{16}$ — giving a bound on the eigenvalues of the Laplace-Beltrami operator of the congruence modular surfaces. For a new method see [BG].

Another preliminary result needed is:

Theorem 3.2. (a) *For a fixed prime* p*, each of the groups* $\mathrm{PSL}_2(p^k)$, $k \in \mathbb{N}$ *and* $p^k > 17$*, has a symmetric subset* S_p *of* $p+1$ *generators for which the Cayley graphs* $X = Cay(\mathrm{SL}_2(p^k), S_p)$ *is a* $(p+1)$*-regular Ramanujan graph, i.e.* $\lambda(X) \leq \frac{2\sqrt{p}}{p+1}$.

(b) *The set of generators* S_p *in part (a) can be chosen to be of the form* $\{h^{-1}Ch \,|\, h \in H\}$*, where* C *is some element of* $\mathrm{SL}_2(p^k)$ *and* H *is a fixed non-split torus of* $\mathrm{PGL}_2(p)$*. (The proof will give a more detailed description of* S_p*).*

Before proving Theorem 3.2, let us mention that part (a) has already been proven by Morgenstern [Mo], but the specific form of the generators as in (b) is crucial for our needs. We therefore apply to [LSV2] instead of [Mo]. We recall the construction there: Let $\mathbb{F}_q$ be the field of order q (a prime power), $\mathbb{F}_{q^d}$ the extension of dimension d and ϕ a generator of the Galois group $\mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$. Fix a basis $\{\xi_0, \ldots, \xi_{d-1}\}$ for $\mathbb{F}_{q^d}$ over $\mathbb{F}_q$ where $\xi_i = \phi^i(\xi_0)$. Extend ϕ to an automorphism of the function field $k_1 = \mathbb{F}_{q^d}(y)$ by setting $\phi(y) = y$; the fixed subfield is $k = \mathbb{F}_q(y)$, of codimension d.

Following the notation in [LSV2], we will denote by R_T the ring $\mathbb{F}_q[y, \frac{1}{1+y}]$ and for every commutative R_T-algebra (with unit) S, we denote by y the element $y\cdot 1\in S$. For such S one defines an S-algebra $A(S)=\bigoplus_{i,j=0}^{d-1} S\xi_i z^j$ with the relations $z\xi_i=\phi(\xi_i)z$ and $z^d=1+y$. Let $R=F_q[y,\frac{1}{y},\frac{1}{1+y}]\subseteq k$ and denote $b=1+z^{-1}\in A(R)$. For every $u\in\mathbb{F}_{q^d}^*\subset A(R)^*$, we denote $b_u=ubu^{-1}$. As $\mathbb{F}_q^*$ is in the center of $A(R), b_u$ depends on the coset of u in $\mathbb{F}_{q^d}^*/\mathbb{F}_q^*$. This gives $\frac{q^d-1}{q-1}$ elements $\{b_u|u\in\mathbb{F}_{q^d}^*/\mathbb{F}_q^*\}$ of $A(R)^*$. The subgroup of $A(R)^*$ generated by the b_u's is denoted $\tilde{\Gamma}$ and its image in $A(R)^*/R^*$ by $\Gamma=\Gamma_{d,q}$. For every ideal $I\lhd R$, we get a map

$$\pi_I : A(R)^*/R^*\to A(R/I)^*/(R/I)^*.$$

The intersection $\Gamma\cap\mathrm{Ker}\pi_I$ is denoted $\Gamma(I)$.

Theorem 6.2 of [LSV2] says:

Theorem 3.3. *For every $d\geq 2$ and every $0\neq I\lhd R$, the Cayley complex of $\Gamma/\Gamma(I)$ is a Ramanujan complex.*

The reader is referred to [LSV1] and [LSV2] for the precise definition of Ramanujan complex and for the precise complex structure of $\Gamma/\Gamma(I)$. What is relevant for us here is that this gives a spectral gap on the Cayley graph of $\Gamma/\Gamma(I)$ with respect to the $\frac{q^d-1}{q-1}$ generators $S=\{b_u|u\in\mathbb{F}_{q^d}^*/\mathbb{F}_q^*\}$.

When $d=2$, S is a symmetric set of generators of Γ and so $Cay(\Gamma/\Gamma(I);S)$ is a $k=(q+1)$-regular graph. When $d\geq 3$, $S\cap S^{-1}=\emptyset$ and $Cay(\Gamma/\Gamma(I);S)$ is a $k=\frac{2(q^d-1)}{q-1}$-regular graph. Let A be its adjacency matrix and $\Delta=\frac{1}{k}A$ the normalized one. Theorem 3.3 implies:

Corollary 3.4. Denote by μ_d-the roots of unity in $\mathbb{C}$ of degree d and $E_d=\{\frac{y+\bar{y}}{2}|y\in\mu_d\}$. Let λ be an eigenvalue of Δ. Then either $\lambda\in E_d$ or $|\lambda|\leq\frac{dq^{(d-1)/2}}{(q^d-1)/(q-1)}$.

Remark 3.5. Note that when $d=2$, $k=|S|=q+1, E_d=\{\pm 1\}$ and Corollary 3.4 states that $Cay(\Gamma/\Gamma(I);S)$ are Ramanujan graphs. The proof of this bound for $d=2$ requires Drinfeld theorem (the Ramanujan conjecture for GL_2 over positive characteristic fields) and for $d\geq 3$ is based on Lafforgue's work [La]. It also requires the Jacquet-Langlands correspondence in positive characteristic (while this correspondense is not fully proved in the literature for $d\geq 3$, we use it here only for $d=2$, which is fully proved — see [LSV1], Remark 1.6). We also mention that for $d\geq 3$, quantitative estimates on Kazhdan property (T) for

$\mathrm{PGL}_d(\mathbb{F}_q((y)))$ can give a weaker estimate such as: either $\lambda \in E_d$ or

$$\lambda \leq \frac{1}{\sqrt{q}} + o(1) \leq \frac{19}{20},$$

which is valid for every d and q. But the case of $d = 2$ needs the deep results from the theory of automorphic forms.

Remark 3.6. The description above of the results from [LSV2] brings only what is relevant to this paper. The bigger picture is as follows: The group $A(R)^*/R^*$ is a discrete cocompact lattice in $A(\mathbb{F}_q((y)))^*/\mathbb{F}_q((y))^*$. The latter is isomorphic to $H = P\mathrm{GL}_d(\mathbb{F}_q((y)))$ and it acts on its Bruhat-Tits building $\mathbb{B}$. The element $b \in H$ takes the initial point of the building (the vertex x_0 corresponding to the lattice $\mathbb{F}_q[[y]]^d$) to a vertex x_1 of distance one from it, where the color of the edge (x_0, x_1) is also one (so x_1 corresponds to an $\mathbb{F}_q[[y]]$-submodule of $\mathbb{F}_q[[y]]^d$ of index q). The group $\mathbb{F}^*_{q^d}/\mathbb{F}^*_q$ acts transitively on these $(q^d - 1)/(q - 1)$ neighbors of x_0 of this type and the group Γ generated by the b_u's acts simply transitively on the vertices of $\mathbb{B}$ — a result which goes back to Cartwright and Steger [CS]. For $b_u \in S$, b_u^{-1} takes x_0 to a neighboring vertex of x_0 where the edge is of color $d - 1$. When $d = 2$, $d - 1 = 1$, and S is a symmetric set of size $q + 1$ and Corollary 3.4 says that $Cay(\Gamma/\Gamma(I), S)$ are Ramanujan graphs. For $d \geq 3$, $S \cap S^{-1} = \emptyset$ and $Cay(\Gamma/\Gamma(I), S)$ are regular graphs of degree $2|S| = \frac{2(q^d-1)}{q-1}$. The Ramanujan complex $\Gamma/\Gamma(I)$ is in fact isomorphic to the quotient $\Gamma(I)\backslash\mathbb{B}$ of the Bruhat-Tits building. On the building $\mathbb{B}$ (and on its quotients $\Gamma(I)\backslash\mathbb{B}$) we have an action of $d - 1$ Hecke operators $A_1, \cdots, A_{d-1}$ and the Ramanujan property gives bounds on their eigenvalues. For $d \geq 3$, $A_1 + A_{d-1}$ is nothing more than the adjacency operator of the Cayley graph of $\Gamma/\Gamma(I)$ with generators SUS^{-1}, and for $d = 2$, $A_1 = A_{d-1}$ and A_1 is the adjacency operator.

The structure of the quotient group $\Gamma/\Gamma(I)$ is analyzed in [LSV2]; If I is a prime ideal of R with $R/I \simeq \mathbb{F}_{q^e}$, then $\Gamma/\Gamma(I)$ is isomorphic to a subgroup of $\mathrm{PGL}_d(q^e)$ containing $\mathrm{PSL}_d(q^e)$. Theorem 7.1 of [LSV2] gives a more precise description, showing that essentially all subgroups between $\mathrm{PSL}_d(q^e)$ and $\mathrm{PGL}_d(q^e)$ can be obtained if I is chosen properly. The image of S in $\mathrm{PGL}_d(q^e)$ which we also denote by S is composed of one element C, the image of b in the notations above, and the conjugates of C by the non-split tori in $\mathrm{PGL}_d(q^e)$ of order $(q^d - 1)/(q - 1)$.

Note that $\mathrm{PGL}_d(q^e)/\mathrm{PSL}_d(q^e)$ is a cyclic group and the image of S there is a single element — the image of C, since all the other elements of S are conjugates of C. The eigenvalues in E_d above, may appear as "lift up" of the eigenvalues of the cyclic group generated by C (which is a subgroup of $\mathrm{PGL}_d(q^e)/\mathrm{PSL}_d(q^e)$ and a quotient of $\Gamma/\Gamma(I)$) whose order divides d. These eigenvalues will be called "the trivial eigenvalues" of $Cay(\Gamma/\Gamma(I);S)$ and they are in the subset E_d defined in Corollary 3.4. If the image of Γ in $\mathrm{PGL}_d(q^e)$ is only $\mathrm{PSL}_d(q^e)$, then 1 is the only trivial eigenvalue of Δ and all the others satisfy the bound of Corollary 3.4.

For d large the issue of which subgroup of $\mathrm{PGL}_d(q^e)$ is obtained is somewhat delicate. For $d = 2$, which is what is needed here for the proof of Theorem 3.2, Theorem 7.1 of [LSV2] ensures that for $p^e > 17, \mathrm{PSL}_2(p^e)$ can be obtained, if I is chosen properly. Thus $Cay(\mathrm{PSL}_2(p^e), S)$ are $(p+1)$-regular Ramanujan graphs. They are, therefore, also ε-expanders by Proposition 2.1(ii), but with an unbounded number of generators when p is going to infinity.

We now show that this is true also with a bounded number of generators. An explicit form of Theorem 1.3 is:

Theorem 3.7. *The family of Cayley graphs $Cay(\mathrm{PSL}_2(\ell); \{A, B, C, C'\})$, when $\ell = p^e$ is any prime power, are uniformly expanders. (Here $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, C is as in the description above when $d = 2$ and $q = p$ and C' will be described in the proof).*

Proof. Let C be as described above (with $d = 2$ and $q = p$ a prime). The image of S in $\mathrm{PSL}_2(p^e)$ as described above, is the set of conjugates of C under the action of the non-split torus T of $\mathrm{PGL}_2(p)$ which is isomorphic to $\mathbb{F}_{p^2}^*/\mathbb{F}_p^*$ and of order $p+1$. Denote $T_1 = T \cap \mathrm{PSL}_2(p)$ a subgroup of index at most 2 in T (in fact index 2, unless $p = 2$). Let C and C' be two representatives of the orbits of S, under conjugation by T_1: C as before and C' a representative of the other orbit (if exists).

We can now prove the Theorem by using Proposition 2.1: Let (V, ρ) be a unitary representation of $\mathrm{PSL}_2(\ell)$, with an $\{A, B, C, C'\}$-almost invariant vector v. Restrict the representation ρ to the subgroup $\mathrm{PSL}_2(p)$. By Theorem 3.1, $Cay(\mathrm{PSL}_2(p); \{A, B\})$ are expanders and hence by Proposition 2.1(iii), v is $\mathrm{PSL}_2(p)$ almost invariant. As it is also C-almost invariant, it is almost invariant under the set $\mathrm{PSL}_2(p) \cdot C \cdot \mathrm{PSL}_2(p)$ and similarly with $\mathrm{PSL}_2(p) \cdot C' \cdot \mathrm{PSL}_2(p)$. The union of these last two sets contain S. So, v is S-almost invariant. But $\lambda(Cay(\mathrm{PSL}_2(p^e), S)) \leq \frac{2\sqrt{p}}{p+1} < \frac{19}{20}$ for every p and e, so by Proposition 2.1(ii), v is $\mathrm{PSL}_2(p^e)$-almost invariant and by Proposition 2.1(i),

$Cay(\mathrm{PSL}_2(p^e); \{A, B, C, C'\})$ are uniform expanders. This finishes the proof of Theorem 3.7 (and hence also of Theorem 1.3). $\square$

Recall that by Proposition 2.3, Theorem 3.7 also says that the family $\{\mathrm{SL}_2(\ell) | \ell \text{ a prime power}\}$ is a family of expanders. Let us now quote:

Theorem 3.8 (Hadad [H1], Theorem 1.2)**.** *Let R be a finitely generated ring with stable range r and assume that the group $EL_d(R)$ for some $d \geq r$ has Kazhdan constant (k_0, ε_0). Then there exist $\varepsilon = \varepsilon(\varepsilon_0) > 0$ and $k = k(k_0) \in \mathbb{N}$ such that for every $n \geq d, EL_n(R)$ has Kazhdan constant (k, ε).*

We refer the reader to [H1] for the proof. We only mention here that if R is a field then its stable range is 1 and $EL_n(R)$, the group of $n \times n$ matrices over R generated by the elementary matrices, is $\mathrm{SL}_n(R)$. Also recall that a finite group G has Kazhdan constant (k, ε) if it has a set of generators S of size at most k, such that for every non-trivial irreducible representation (V, ρ) of G and for every $0 \neq v \in V$, there exists $s \in S$ such that $\|\rho(s)v - v\| \geq \varepsilon \|v\|$. As is well known, this implies that $Cay(G; S)$ is ε'-expander for some ε' which depends only on ε. All these remarks combined with Theorems 3.7 and 3.8 give:

Theorem 3.9. *The groups $\left\{\mathrm{SL}_n(q) \;\middle|\; \text{all } \ 2 \leq n \in \mathbb{N},\ q \ \text{ prime power}\right\}$ form a family of expanders uniformly.*

Remark 3.10. In the proof of Theorem 3.9, we used for $d \geq 3$, Theorem 3.8 of Hadad whose proof was heavily influenced by Kassabov's proof [K1] that all $\mathrm{SL}_n(q), n \geq 3$, are expanders. So our proof cannot be considered as a really different proof for $n \geq 3$. In [KLN], a second very different proof for $\mathrm{SL}_n, n \geq 3$ was announced, based on the theory of Ramanujan complexes. But it turns out that the proof sketched there has a mistake, for which the author of the current paper takes full responsibility. The idea there was to handle SL_d, d even, say $d = 2m$, by using the following argument: Corollary 3.4 above gives a spectral gap with respect to an unbounded subset S of $\mathrm{(P)GL}_d(q)$ which consists of conjugates of a single element by a non-split tori T. This T as a subgroup of $G = \mathrm{GL}_d(q)$ is inside a copy of $H = \mathrm{GL}_2(q^m)$. Passing from G to $\mathrm{PGL}_d(q)$, T is then in the image $\bar{H}$ of H. We argued there that by dividing by the center, $T \subset \bar{H}$ and $\bar{H}$ is isomorphic to $\mathrm{PGL}_2(q^m)$. (We then wanted to use Theorem 1.3 for $\bar{H}$ to deduce that $\bar{H}$ is an expander and to continue to argue as in the proof of Theorem 3.7). It is not true however, that $\bar{H}$ is $\mathrm{PGL}_2(q^m)$: we divided by the center of G which is of order at most d and not by the center of H which is of order $q^m - 1 >> d$. So $\bar{H}$ has a large abelian quotient and it is far from being an expander.

4. Bounded generation by SL(2)

A finite group G is said to be a product of s copies of SL_2, if there exist prime powers q_i and homomorphisms $\varphi_i : \mathrm{SL}_2(q_i) \to G$, $i = 1, \ldots, s$, such that for every $g \in G$ there exist $x_i \in \mathrm{SL}_2(q_i)$, $i = 1, \ldots, s$ with $g = \varphi_1(x_1) \cdot \ldots \cdot \varphi_s(x_s)$.

Theorem 3.7 shows that all the groups $\mathrm{SL}_2(q)$ are uniform expanders (with 4 generators for each one). It now follows from Corollary 2.2 that for a fixed s, all the groups which are products of s copies of SL_2 are uniform expanders with $4s$ generators. We will now show that this is indeed the case for all finite simple groups of Lie type of bounded rank, excluding the groups of Suzuki type.

Theorem 4.1. *There exists a function $f : \mathbb{N} \to \mathbb{N}$, such that if G is a finite simple group of Lie type of rank r, but not of Suzuki type, then it is a product of $f(r)$ copies of* SL_2.

Before giving the proof we remark that Theorem 4.1 combined with Theorem 3.9 and the result of Nikolov [N] implies Theorem 1.1. Indeed, by [N], a classical group of Lie type is a bounded product of groups of type $\mathrm{SL}_n(q)$ (n and q varies) and so by Theorem 3.9 they are uniform expanders. The other finite simple groups of Lie type have bounded rank and so are bounded product of SL_2 by Theorem 4.1, and hence also uniform expanders. This excludes, of course, the Suzuki groups to which every homomorphism from $\mathrm{SL}_2(q)$ has trivial image, since their order is not divisible by 3. Thus the validity of Theorem 1.2 for the Suzuki groups is left open.

Back to Theorem 4.1. This result has been announced in [KLN] and a model theoretic proof based on the work of Hrushovski and Pillay [HP] was sketched there. Recently, Liebeck, Nikolov and Shalev [LNS] proved the theorem by standard group theoretic arguments. This is somewhat more technical and requires some case by case analysis but has the advantage that they came out with an explicit function $f(r)$ which is valid for every group G of rank r. This is of importance for our application for expanders as it enables one to deduce explicit k and ε in Theorem 1.1.

Anyway, we will bring here the model theoretic proof. For a nice introduction to the model theory of finite simple groups, see [W]. As there are only finitely many group types of bounded rank, we can take G to be a fixed (twisted or untwisted) Chevalley group and we need to prove the result for the groups $G(F)$ when F is a finite field. We will show below that each such $G(F)$ contains a copy of $(\mathrm{P})\mathrm{SL}_2(F)$ as a *uniformly definable subgroup*. By a definable subgroup, we mean a subgroup that can be defined using a first order sentence in the language of rings with a distinguished endomorphism — the language in which G is defined. By uniformly definable we mean that the subgroup $(\mathrm{P})\mathrm{SL}_2(F)$ is defined by a single sentence — independent of F.

Assuming this fact, we can argue as follows: Let F_i be an infinite family of finite fields and $K = (\Pi F_i)/U$ an ultra product of them, i.e., U is a non-principal ultra-filter. Thus K is a pseudo-algebraically closed field (PAC, for short — see [FJ] and [HP]). Let $\tilde{G} = \Pi G(F_i)/U$ the corresponding ultra product of the groups $G(F_i)$. By a basic result (Point [P], Propositions 1 and 2 and Corollary 1) $\tilde{G}$ is a simple group isomorphic to $G(K)$ and similarly the ultra-product of the $(\mathrm{P})\mathrm{SL}_2(F_i)$'s gives a subgroup $(\mathrm{P})\mathrm{SL}_2(K)$ of $\tilde{G} = G(K)$.

Now, as $\tilde{G} = G(K)$ is simple, it is generated by the conjugates of $(\mathrm{P})\mathrm{SL}_2(K)$. By [HP, Proposition 2.1] $G(K)$ is a product of $m < \infty$ conjugates of $(\mathrm{P})\mathrm{SL}_2(K)$. This is an elementary statement about $G(K)$ and hence it is true also for $G(F_i)$ for almost all i. This proves what we need modulo the promised fact.

Remark 4.2. The model theoretic proof gives (when one follows the arguments in [HP]) that $m \leq 4 \dim G$. Moreover, in principal one can give an explicit bound M such that the above claim is true for every F with $|F| > M$. The proof in [LNS] gives explicit bounds on m which are usually (but not always) slightly better and are valid for all F.

We are left with proving our claim that $G(F)$ contains a copy of $(\mathrm{P})\mathrm{SL}_2(F)$ as a uniformly definable subgroup.

If G splits (i.e. untwisted type), e.g. $G = E_6$, it contains SL_2 as a subgroup generated by a root subgroup and its opposite. Note that a root subgroup is definable and as SL_2 is a bounded product of the root subgroup and its opposite, it is also definable. Of course, in this case it is even an algebraic subgroup and a copy of $(\mathrm{P})\mathrm{SL}_2(F)$ in $G(F)$ can be defined by polynomials independent of F.

If G is twisted, but not a group of Ree type (i.e. all the simple roots of G are of the same length, so the type is A_n, D_n or E_6), e.g., look at $G(q) = {}^2E_6(q)$. Then G is the group of points of $E_6(q^2)$ of the following form: $\{g \in E_6(q^2) \mid g^{F_r} = g^{\tau}\}$ where τ is the graph automorphism of E_6 and F_r the Frobenius automorphism. By restriction of scalars, this is an algebraic group defined over $\mathbb{F}_q$. If the automorphism τ has a fixed vertex, e.g. for our example ${}^2E_6$, then ${}^2E_6(q)$ contains a copy of $\mathrm{SL}_2(q)(\subseteq \mathrm{SL}_2(q^2) \subseteq E_6(q^2))$ corresponding to this vertex and as an $\mathbb{F}_q$-group — this is an algebraic subgroup. The argument we illustrated here with ${}^2E_6(q)$ works equally well with the other twisted groups with fixed vertex (of course, for ${}^3D_4$ we should take $D_4(q^3)$ — but the rest is the same). By the well known classification of the simple algebraic groups over finite fields, we have covered all cases except for the twisted forms of A_n, n even. But ${}^2A_n$ is anyway $SU(n+1)$ which contains $SU(2)$ (as uniformly definable algebraic subgroup) and it is well known that $SU(2,q^2)$ is isomorphic to $\mathrm{SL}_2(q)$.

We are left with the twisted groups of Ree type: ${}^2F_4(2^{2n+1})$ and ${}^2G_2(3^{2n+1})$ (the other type ${}^2B_2(2^{2n+1})$ give the Suzuki groups and these were excluded from the theorem). Now, ${}^2F_4(2^{2n+1})$ is known (cf. [GLS] Table 2.4 VI, Table 2.4.7, Theorems 2.4.5 and 2.48) to have a subgroup generated by a root subgroup and its opposite which is isomorphic to $\mathrm{SL}_2(2^{2n+1})$. (This is not the case for all roots; for some we get the Suzuki groups, but we need only one root which gives SL_2). For ${}^2G_2(3^{2n+1})$ one can argue by pure group theoretical terms: it is known (cf. [E]) to have a unique conjugacy class of involutions and if τ is such an involution, then $C_G(\tau)$ — the centralizer of τ — is isomorphic to $H = \langle\tau\rangle \times \mathrm{PSL}_2(3^{2n+1})$. Within $H, \mathrm{PSL}_2(3^{2n+1})$ is the set of all commutators of H (since every element of $\mathrm{PSL}_2(q)$ is a commutator). Thus $\mathrm{PSL}_2(3^{2n+1})$ is a uniformly definable subgroup of ${}^2G_2(3^{2n+1})$. The proof of Theorem 4.1 (and hence of 1.1) is now complete.

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